SNUTP 94-110
hep-th/9502114

Stability Analysis of the Dilatonic Black Hole in Two Dimensions

Won Tae Kim[(a)] and Julian Lee[(b)]
Center for Theoretical Physics and Department of Physics,
Seoul National University, Seoul 151-742, Korea

Young-Jai Park[(c)]
Department of Physics and Basic Science Research Institute,
Sogang University, C.P.O. Box 1142, Seoul 100-611, Korea

ABSTRACT

We explicitly show that the net number of degrees of freedom in the two-dimensional dilaton gravity is zero through the Hamiltonian constraint analysis. This implies that the local space-time dependent physical excitations do not exist. From the linear perturbation around the black hole background, we explicitly prove that the exponentially growing mode with time is in fact eliminated outside the horizon. Therefore, the two-dimensional dilation gravity is essentially stable.

[(a)] E-mail address : wtkim@phyb.snu.ac.kr
[(b)] E-mail address : lee@phyb.snu.ac.kr
[(c)] E-mail address : yjpark@ccs.sogang.ac.kr

A black hole solution to the two-dimensional critical string theory has attracted much interest [1]. It is given by the solution of the modular invariant $SL(2,R)/U(1)$ gauged Wess-Zumino-Witten(WZW) coset model of a conformal field theory. The black hole solution can be also derived by solving the two-dimensional beta-function equations of the string theory, which is effectively described by the two-dimensional dilaton gravity at the Lagrangian level [2]. This theory can be used as a toy model in two dimensions which can be the basic starting point in resolving the interesting puzzles of gravitational system such as the end point of Hawking radiation and information loss problem inside the black hole [3,4]. Then it becomes natural task to investigate the classical stability of the black hole solution for this model. To determine whether the black hole is stable or not, the small (linear) perturbation of the classical equations of motion in the black hole background was considered in the regular region, which usually sees the effective potential [5]. As is well known, if there exists an exponentially growing mode with time, the black hole is unstable [6]. At first sight, one can understand that there should be no physical instability, because one can easily see that the degrees of freedom in the two-dimensional dilaton gravity is zero in the absence of matter fields. However it was claimed that the growing mode with time in the gravity sector exists [7]. Therefore, their claim amounts to the statement that the black hole is unstable in the absence of matter fields. More recently, it was conjectured that the exponentially growing mode with time induced by potential well should be a gauge artifact simply by counting degrees of freedom [8] for the gravitational field which is given by $\mathrm{d}(\mathrm{d}-3)/2$ in d dimensions [9].

In this paper, we reconsider the two-dimensional dilaton gravity to clarify these points and elaborate whether the black hole is really stable or not. In particular, we will show that the two-dimensional static black hole solution is stable against linear perturbation since the unstable mode related to the global symmetry of the theory can be eliminated by an appropriate global coordinate transformation. As a guidance, we first enumerate the net number of degrees of freedom in terms of Hamiltonian constraint analysis where it turns out to be a null degree. It implies that no local propagating modes exist. From this point of view, we try to find what is the possible perturbation of black hole solution, and show that the only possible perturbations correspond to the

variations of the three global parameters, one is the black hole mass and the others are the position coordinates of black hole. Next we shall apply the conventional linear perturbation method around the black hole background to explicitly see the behavior of the classical perturbation modes. We then identify the growing mode with time with the variations of the global parameters by comparing the above results. We conclude that the exponentially growing mode with time is eliminated by choosing a suitable position coordinate, which means that the two-dimensional dilatonic black hole is stable against the linear perturbation.

Let us now start with the two-dimensional dilaton gravity defined by [3]

$$S_D = \int d^2x \sqrt{G} e^{-2\phi} [R + 4g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + 4\lambda^2], \tag{1}$$

where $G = -\det g_{\mu\nu}$ and ϕ is a dilaton field, and λ is a cosmological constant.

Following the Arnowitt-Deser-Misner(ADM) formulation, the two-dimensional $g_{\mu\nu}$ is written by

$$g_{\mu\nu} = \gamma \begin{pmatrix} -N_0^2 + N_1^2 & N_1 \\ N_1 & 1 \end{pmatrix}, \tag{2}$$

where N_0 and N_1 are lapse and shift functions respectively, and we factor out the conformal factor γ [10].

Then the action (1) can be rewritten by the first order form as follows

$$S_D = \int d^2x (\pi_\phi \partial_0 \phi + \pi_\gamma \partial_0 \gamma - N_0 G_0 - N_1 G_1), \tag{3}$$

where the generators of reparametrization are

$$\begin{aligned} G_0 &= 2\gamma e^{2\phi} (\pi_\phi + 2\gamma \pi_\gamma) \pi_\gamma - e^{-2\phi} ((\partial_1 \phi)^2 + \lambda^2), \\ G_1 &= -\pi_\gamma \partial_1 \gamma + \pi_\phi \partial_1 \phi, \end{aligned} \tag{4}$$

and π_ϕ, π_γ are canonical momenta with respect to the dilaton and conformal factor γ respectively. Two primary constraints for auxiliary fields are π_{N_0} and π_{N_1}, which are fully first class constraints [11]. Two secondary constraints G_0, G_1 corresponding to the Virasoro constraints satisfy the closed algebra under Poisson brackets,

$$\begin{aligned} \{G_0(x), G_0(y)\} &= (G_1(x) + G_1(y)) \partial_1 (x^1 - y^1), \\ \{G_1(x), G_1(y)\} &= (G_1(x) + G_1(y)) \partial_1 (x^1 - y^1), \\ \{G_0(x), G_1(y)\} &= (G_0(x) + G_0(y)) \partial_1 (x^1 - y^1). \end{aligned} \tag{5}$$

Therefore, the true physical degree of freedom is zero because the graviton and dilaton degrees of freedom are four, while there are four first-class constraints, *i.e.*, $(3+1)-(2+2)=0$. As a natural result, the local excitations are absent. However, there are still global degrees of freedom, which will be considered later.

The equations of motion for the graviton and dilaton are given by

$$\begin{aligned} R_{\mu\nu} &+ 2\nabla_\mu\nabla_\nu\phi = 0, \\ \Box\phi &- 2(\nabla\phi)^2 + 2\lambda^2 = 0. \end{aligned} \tag{6}$$

The classical theory is most easily analyzed in the conformal gauge as follows

$$ds^2 = 2g_{+-}dx^+dx^-, \qquad g_{+-} = -\frac{1}{2}e^{2\rho(x^+,x^-)}, \tag{7}$$

where $x^\pm = x^0 \pm x^1$, and the non-vanishing Christoffel symbols are $\Gamma^+_{++} = 2\partial_+\rho$ and $\Gamma^-_{--} = 2\partial_-\rho$. In the conformal gauge, we have the equation of motion

$$\partial_-\partial_+(\rho - \phi) = 0, \tag{8}$$

which yields

$$\rho(x) = \phi(x) + \frac{1}{2}(w_+(x^+) + w_-(x^-)) \tag{9}$$

with arbitrary functions $w_\pm(x^\pm)$. However, in conformal gauge there is still a residual gauge transformation of the form

$$\rho'(x') = \rho(x) - \frac{1}{2}\left(\ln(\frac{\partial x'^+}{\partial x^+}) + \ln(\frac{\partial x'^-}{\partial x^-})\right). \tag{10}$$

Therefore we can fix the residual gauge symmetry such that the functions $w_\pm$ vanishes. Our calculations will be done in this Kruskal gauge. This gauge is quite a rigid one and only possible coordinate transformation, which preserves the condition (8), is the global transformation $x^\pm \to x^\pm + a^\pm$ with some constant $a^\pm$, which is just a translation. We will see later that the exponentially growing modes are directly related to it. In this gauge, the black hole solution is given by [3]

$$e^{-2\bar\rho(x^+,x^-)} = e^{-2\bar\phi(x^+,x^-)} = \frac{m}{\lambda} - \lambda^2(x^+ - x_0^+)(x^- - x_0^-), \tag{11}$$

where m is an integration constant which turns out to be a black hole mass [1].

Note that the event horizon of the black hole solution (11) is $x_H^+ = x_0^+$ or $x_H^- = x_0^-$. The arbitrariness is due to the above mentioned translational symmetry, and we may freely set the black hole position $x_0^\pm = 0$ by a global transformation $x^\pm \to x^\pm + x_0^\pm$ in eq. (11). This black hole solution (11) is exact without any approximations because differential equations (6) are exactly solvable. Furthermore, the system has no local degrees of freedom, and the black hole solution describes the global geometry. In this two-dimensional dilatonic gravity, the only possible perturbation around the static background is due to the variation of three global parameters, which are black hole mass m and the black hole position $x_0^\pm$. To make it explicit, we can vary these parameters in (11) to get

$$\begin{aligned}
\phi(x^+ &, x^-; m+\delta m, x_0^\pm + \delta x_0^\pm) - \bar{\phi}(x^+, x^-; m, x_0^\pm) \\
&= \delta m \frac{\partial}{\partial m}\bar{\phi}(x^+, x^-; m, x_0^\pm) \\
&+ \delta x_0^+ \frac{\partial}{\partial x_0^+}\bar{\phi}(x^+, x^-; m, x_0^\pm) + \delta x_0^- \frac{\partial}{\partial x_0^-}\bar{\phi}(x^+, x^-; m, x_0^\pm) + \cdots \\
&= \frac{-\frac{\delta m}{2\lambda} - \frac{\lambda^2}{2}(\delta x_0^- x^+ + \delta x_0^+ x^-)}{\left[\frac{m}{\lambda} - \lambda^2 (x^+ - x_0^+)(x^- - x_0^-)\right]} + \cdots,
\end{aligned} \tag{12}$$

where the dots mean higher order terms. This relation will be used later to identify the perturbation modes.

Let us now study a small perturbation in terms of the conventional linear perturbation method around the classical black hole solution [5]. The result should be consistent with eq. (12) if we identify some integration constants, which arise from solving the perturbed differential equations, with known quantities. The linear perturbed fields around the static black hole solution $(\bar{g}_{\mu\nu}, \bar{\phi})$ are defined by

$$\begin{aligned}
g_{\mu\nu}(x^+, x^-) &= \bar{g}_{\mu\nu}(x^+, x^-) + h_{\mu\nu}(x^+, x^-), \\
\phi(x^+, x^-) &= \bar{\phi}(x^+, x^-) + \delta\phi(x^+, x^-).
\end{aligned} \tag{13}$$

Then, the linear perturbation leads to the perturbed equations of motion as follows

$$\begin{aligned}
&\delta R_{\mu\nu} + 2(\bar{\nabla}_\mu \bar{\nabla}_\nu \delta\phi - \delta\Gamma^\alpha_{\mu\nu}\bar{\nabla}_\alpha \bar{\phi}) = 0, \\
&\quad h^{\mu\nu}\ (-\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\phi} + 2\bar{\nabla}_\mu \bar{\phi} \bar{\nabla}_\nu \bar{\phi}) + \bar{g}^{\mu\nu}(\bar{\nabla}_\mu \bar{\nabla}_\nu \delta\phi - 4\bar{\nabla}_\mu \bar{\phi}\bar{\nabla}_\nu \delta\phi - \delta\Gamma^\alpha_{\mu\nu}\bar{\nabla}_\alpha \bar{\phi}) = 0,
\end{aligned} \tag{14}$$

where the upper bars represent the background quantities, and $\delta R_{\mu\nu}$, and $\delta\Gamma^{\alpha}_{\mu\nu}$ are given by

$$\begin{aligned}\delta R_{\mu\nu} &= \frac{1}{2}(-\bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h^{\alpha}_{\alpha} + \bar{\nabla}^\alpha \bar{\nabla}_\mu h_{\nu\alpha} + \bar{\nabla}^\alpha \bar{\nabla}_\nu h_{\mu\alpha}),\\ \delta\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}\bar{g}^{\alpha\beta}(\bar{\nabla}_\mu h_{\nu\beta} + \bar{\nabla}_\nu h_{\mu\beta} - \bar{\nabla}_\beta h_{\mu\nu}).\end{aligned} \tag{15}$$

We may consider only the variation of conformal factor since the metric can be always written in conformally flat form where $h_{\mu\nu}(x^+, x^-) = 2\delta\rho(x^+, x^-)\bar{g}_{\mu\nu}(x^+, x^-)$ in two dimensions. Also we keep ourselves in the Kruskal gauge, which tells us that

$$\delta\rho(x^+, x^-) = \delta\phi(x^+, x^-). \tag{16}$$

Note that the classical background solution (11) also satisfies the condition $\bar{\rho} = \bar{\phi}$ since the residual gauge symmetry can be fixed in the conformal gauge [3]. For the perturbed fields, the same condition holds, and the full solution $(\bar{\phi} + \delta\phi)$ will be meaningful within the same gauge fixing condition. In this gauge, the linearized equation (14) can be written as

$$\partial_+^2 \delta\phi \; - \; 4\partial_+\bar{\phi}\partial_+\delta\phi = 0, \tag{17}$$

$$\partial_-^2 \delta\phi \; - \; 4\partial_-\bar{\phi}\partial_-\delta\phi = 0, \tag{18}$$

$$\partial_+\partial_- \; \delta\phi \; -2\partial_+\bar{\phi}\partial_-\delta\phi - 2\partial_-\bar{\phi}\partial_+\delta\phi - 2(\partial_+\partial_-\bar{\phi} - 2\partial_+\bar{\phi}\partial_-\bar{\phi})\delta\phi = 0, \tag{19}$$

where eqs.(17) and (18) are constraint equations, and eq. (19) is a perturbed dilaton equation.

To obtain perturbed solutions, we transform equations of motion (17)-(19) from the Kruskal coordinates (x^+, x^-) to the tortoise coordinates (t, r^*). In the (t, r^*) coordinates, we can easily consider non-singular region outside the black hole, and treat time t explicitly. The transformation is given by

$$\begin{aligned} x^+ - x^+_0 &= +\frac{1}{\lambda}e^{+\lambda(t+r^*)} > 0,\\ x^- - x^-_0 &= -\frac{1}{\lambda}e^{-\lambda(t-r^*)} < 0,\end{aligned} \tag{20}$$

where r^* ranges from $-\infty$ to $+\infty$, and the even horizon is at $r^* \to -\infty$, and the infinity $r^* \to +\infty$ corresponds to the asymptotically flat region. We assumed that λ

is positive for simplicity. Then, we rewrite the eqs. (17)-(19) in (t, r^*) coordinates as

$$(\partial_t + \partial_{r^*})^2 \delta\phi - 2\lambda(\partial_t + \partial_{r^*})\delta\phi + \frac{4\lambda}{(1 + \frac{m}{\lambda} e^{-2\lambda r^*})}(\partial_t + \partial_{r^*})\delta\phi = 0, \tag{21}$$

$$(\partial_t - \partial_{r^*})^2 \delta\phi + 2\lambda(\partial_t - \partial_{r^*})\delta\phi + \frac{4\lambda}{(1 + \frac{m}{\lambda} e^{-2\lambda r^*})}(-\partial_t + \partial_{r^*})\delta\phi = 0, \tag{22}$$

$$(-\partial_t^2 + \partial_{r^*}^2)\delta\phi + \frac{4\lambda}{(1 + \frac{m}{\lambda} e^{-2\lambda r^*})}\partial_{r^*}\delta\phi + \frac{4\lambda^2}{(1 + \frac{m}{\lambda} e^{-2\lambda r^*})}\delta\phi = 0. \tag{23}$$

By adding and subtracting eqs. (21) and (22), the following simplified relations are given,

$$\begin{aligned} (\partial_t^2 + \partial_{r^*}^2)\delta\phi + 2U(r^*)\partial_r^*\delta\phi &= 0, & (24) \\ \partial_t \partial_r^* \delta\phi + U(r^*)\partial_t \delta\phi &= 0, & (25) \end{aligned}$$

where $U(r^*) \equiv \lambda \left(\frac{1 - \frac{m}{\lambda} e^{-2\lambda r^*}}{1 + \frac{m}{\lambda} e^{-2\lambda r^*}} \right)$.

The constraint equation (25) is easily solved with two unknown functions as

$$\begin{aligned} \delta\phi(t\,, r^*) &= A(r^*) + b(t) e^{-\int^{t^*} dr^* U(r^*)}, \qquad (26) \\ A(r^*) &= e^{-\int^{r^*} dr^* U(r^*)} \left(\int^{r^*} \{ a(r^*) e^{\int^{r^*} dr^* U(r^*)} \} dr^* \right), \\ \int^{r^*} dr^* \, U(r^*) &= \lambda r^* + \left(1 + \frac{m}{\lambda} e^{-2\lambda r^*} \right), \end{aligned}$$

where $a(r^*)$ and $b(t)$ are integration functions in solving the differential equation. Interestingly the perturbed dilaton solution is composed of two functions; one is dependent only on the space coordinate, and the other is space-time dependent part which is reminiscent of some propagating modes.

For more details, plugging eq. (26) into the dilaton equation (23) and eq. (24), we obtain two separated ordinary differential equations,

$$\begin{aligned} \frac{d^2}{dr^{*2}} A(r^*) + 2U(r^*) \frac{d}{dr^*} A(r^*) &= 0, & (27) \\ \frac{d^2}{dt^2} b(t) - \lambda^2 b(t) &= 0. & (28) \end{aligned}$$

These equations yield exact solutions as follows,

$$
\begin{aligned}
A(r^*) &= d - \frac{ce^{-2\lambda r^*}}{(1+\frac{m}{\lambda}e^{-2\lambda r^*})}, &(29)\\
b(t) &= \alpha^- \ e^{\lambda t} + \alpha^+ \ e^{-\lambda t}, &(30)
\end{aligned}
$$

where c, d, and $\alpha^\pm$ are space-time independent constants. From the boundary condition $A(\infty)=0$ for the asymptotically flatness, and we set $d=0$. Note that in ref. [7,8], authors have qualitatively considered second part $b(t)$ without amplitudes in our general solution (26) and neglected the time-independent part $a(r^*)$ (or $A(r^*)$) which arise from the integration with respect to time in eq. (25).

Then, the linear perturbation of dilaton (graviton) field is given by

$$
\delta\phi(t,r^*) = \frac{-ce^{-2\lambda r^*} + \alpha^- \ e^{\lambda(t-r^*)} + \alpha^+ \ e^{-\lambda(t+r^*)}}{(1+\frac{m}{\lambda}e^{-2\lambda r^*})}. \quad (31)
$$

At this stage, one might think that the black hole solution is unstable against the time-dependent linear perturbation since the perturbed dilaton solution has an exponentially growing mode with respect to time. However, this is not the case because they corresponds to the variation of the black hole position, which can be eliminated by taking an appropriate coordinate as will be shown below. In the Kruskal coordinates, the solution (31) is written as

$$
\begin{aligned}
\delta\phi(x^+,x^-) &= \delta\rho(x^+,x^-)\\
&= \frac{-(c+\lambda\alpha^- x_0^+ - \lambda\alpha^+ x_0^-) + \lambda\alpha^- \ x^+ - \lambda\alpha^+ \ x^-}{\left[\frac{m}{\lambda} - \lambda^2(x^+ - x_0^+)(x^- - x_0^-)\right]}. &(32)
\end{aligned}
$$

Note that the linear perturbation is compatible with eq. (12) by comparing eqs. (32) and (12), we then identify

$$
\begin{aligned}
c &= \frac{\delta m}{2\lambda} + \frac{\lambda^2}{2}\left(\delta x_0^+ x_0^- + \delta x_0^- x_0^+\right), &(33)\\
\alpha^\pm &= \pm\frac{\lambda}{2}\delta x_0^\pm. &(34)
\end{aligned}
$$

This identification shows that the physical origin of unknown constants $\alpha^\pm$ in front of the growing mode with time is due to the variation of position coordinates of the black hole.

Furthermore, directly substituting the transformation rule (20) into the black hole solution (11), eq. (12) can be written as

$$\delta\phi(t, r^*) = \frac{-\frac{\delta m}{2\lambda} e^{-2\lambda r^*}}{(1 + \frac{m}{\lambda} e^{-2\lambda r^*})} \tag{35}$$

only in terms of the mass parameter. This is because the variation of black hole position is zero in the tortoise coordinate. Only when we allow a constant shift as $x_0^\pm \to x_0^\pm + a^\pm$ in eq. (20), then the growing modes with time may exist in tortoise coordinates. However, the black hole position (or variation of it) are not gauge invariant quantity and just a location of black hole position which we could arbitrarily choose. For this reason, $\alpha^\pm$ can be zero, and the growing modes can be eliminated in the perturbed solution (31) and (32).

In summary, we have explicitly shown that the true local physical degrees of freedom in the two-dimensioal dilaton gravity is zero through the Hamiltonian constraint analysis. Next, we have shown that in the two-dimensional dilaton gravity the only possible physical perturbation (deformation) of the classical background black hole is for the mass parameter, which is the global degree of freedom, and the other two parameters are just a position of black hole, which is not a gauge invariant physical quantity. We have also considered from the direct linear perturbation around the static black hole, where we have obtained the explicit solution of the linearized equations of motion and shown that the amplitudes in front of the exponentially growing solution with time can be eliminated by identifying them with the variation of black hole positions. For these reasons, the two dimensional dilatonic black hole is stable.

Acknowledgements

We would like to thank C. Lee for helpful discussions and encouragement. W. T. Kim and J. Lee were supported in part by the Korea Science and Engineering Foundation through the Center for Theoretical Physics (1994). The present study was also supported in part by the Basic Science Research Institute Program, Ministry of Education, 1994, Project No. 2414.

References

[1] E. Witten, Phys. Rev. D 44 (1991) 314.

[2] G. Mandal, A. Sengupta, and S. R. Wadia, Mod. Phys. Lett. A 6 (1991) 1685; S.P. deAlwis and J. Lykken, Phys. Lett. B 269 (1991) 464; A. Dhar, G. Mandal and S. R. Wadia, Mod. Phys. Lett. A 7 3703 (1992); S. R. Das, Mod. Phys. Lett. A 8 (1993) 69; S. K. Ramma, Phys. Rev. Lett. 70 (1993) 3186 .

[3] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D45 (1992) R1005.

[4] J. G. Russo, L. Susskind, and L. Thorlacius, Phys. Lett. B292 (1991) 13.

[5] T. Regge and J. A. Wheeler, Phys. Rev. 108 (1957) 1403; C. V. Vishveshwara, Phys. Rev. D1 (1970) 2870; F. J. Zerilli, Phys. Rev. D24 (1970) 737.

[6] For a review: S. Chandrasekhar, in: Space and Geometry, eds. R. A. Matzner and L. C. Shepley (Texas University Press, Austin, TX, 1982).

[7] J. Y. Kim, H. W. Lee, and Y. S. Myung, Phys. Lett. B328 (1994) 291.

[8] Y. S. Myung, Phys. Lett. B334 (1994) 29.

[9] S. Weinberg, in: General Relativity, eds, S. W. Hawking and W. Israel (Cambridge Univ. Press, London, 1979).

[10] A. J. Hanson, T. Regge, and C. Teitelboim, "Constrained Hamiltonian Systems", (Accademica Nazionale dei Lincei, Rome, Italy, 1987).

[11] M. Henneaux and C. Teitelboim, in: Quantization of Gauge System (Princeton University Press, Princeton, New Jersey, 1992).